1) Expand the following expressions involving Kronecker deltas, and simplify where possible.

(a) $\delta_{ij}\ \delta_{ij}$, (b) $\delta_{ij}\delta_{jk}\delta_{ki}$, (c) $\delta_{ij}\delta_{jk}$, (d) $\delta_{ij}\ A_{ik}$

Answer: (a) 3, (b) 3, (c) δ_{ik}, (d) A_{jk}

2) If $a_i = \varepsilon_{ijk} b_j c_k$ and $b_i = \varepsilon_{ijk}\ g_j\ h_k$, substitute b_j into the expression for a_i to show that

$$a_i = g_k\ c_k\ h_i - h_k\ c_k\ g_i$$

or, in symbolic notation, $\mathbf{a} = (\mathbf{c} \cdot \mathbf{g})\mathbf{h} - (\mathbf{c} \cdot \mathbf{h})\mathbf{g}$.

3) By summing on the repeated subscripts determine the simplest form of

(a) $\varepsilon_{3jk} a_j a_k$ (b) $\varepsilon_{ijk}\ \delta_{kj}$ (c) $\varepsilon_{1jk}\ a_2 T_{kj}$ (d) $\varepsilon_{1jk}\delta_{3j} v_k$

Answer: (a) 0, (b) 0, (c) $a_2(T_{32} - T_{23})$, (d) $-v_2$

4) (a) Show that the tensor $B_{ik} = \varepsilon_{ijk}\ v_j$ is skew-symmetric.

(b) Let B_{ij} be skew-symmetric, and consider the vector defined by $v_i = \varepsilon_{ijk} B_{jk}$ (often called the *dual* vector of the tensor **B**). Show that $B_{mq} = \frac{1}{2}\varepsilon_{mqi} v_i$.

5) If $A_{ij} = \delta_{ij}\ B_{kk} + 3\ B_{ij}$, determine B_{kk} and using that solve for B_{ij} in terms of A_{ij} and its first invariant, A_{ii}.

Answer: $B_{kk} = \frac{1}{6} A_{kk}$; $B_{ij} = \frac{1}{3} A_{ij} - \frac{1}{18}\delta_{ij} A_{kk}$

6) Show that the value of the quadratic form $T_{ij} x_i x_j$ is unchanged if T_{ij} is replaced by its symmetric part, $\frac{1}{2}(T_{ij} + T_{ji})$.

7) Show by direct expansion (or otherwise) that the box product $\lambda = \varepsilon_{ijk}\ a_i b_j c_k$ is equal to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$