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LEFT-LOOKING VERSION OF *RIF* PRECONDITIONER WITH COMPLETE PIVOTING STRATEGY

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Abstract. In this paper, we use a complete pivoting strategy for the left-looking version of *RIF* preconditioner and study effect of this pivoting.

1. Introduction

Consider the linear system of equations of the form $Ax = b$ where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. Krylov subspace methods can be used to solve this system [3]. An implicit preconditioner M for this system is an approximation of matrix A, *i.e.*, $M \approx A$. This preconditioner can be used as the right preconditioner for this system. In this case, instead of solving the original system $Ax = b$, it is better to solve the right preconditioned system $AM^{-1}u = b$, where $x = M^{-1}u$, by the Krylov subspace methods. *ILU* preconditioners are examples of implicit preconditioners. These type of preconditioners are in the form of $M = LDU$ where L and U^T are unit lower triangular matrices and

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D is a diagonal matrix. In this paper, we propose pivoting for the left-looking version of RIF which is an implicit preconditioner.

2. Complete pivoting strategy for the left-looking version of RIF preconditioner

Suppose that no dropping is applied in Algorithm 1. Also suppose that Π_k and Σ_k, for $1 \leq k \leq i-1$, are the computed row and column permutation matrices at steps 1 to $i-1$ of this algorithm. At the beginning of step i of this algorithm, at first, lines 2 and 3 are set. Then, the internal *while* loop is run and in line 6 the parameter $iter$ is set equal to $iter+1$. After that, vector $z_i^{(i-1)}$ is computed in lines 7-11. The pivot element $q_i^{(i-1)}$ is computed in line 12. The essential relation $q_j^{(i-1)} = e_j^T(\Pi A \Sigma) z_i^{(i-1)}$, for $j \geq i+1$, gives the chance to compute the vector $(q_{i+1}^{(i-1)}, \cdots, q_n^{(i-1)})$ in lines 13-15. This vector is used to apply the row pivoting strategy in lines 16-22 and matrix Π is updated. Since the balance of the column pivoting has been ruined, then $satisfied\_p$ is set to false in line 18. After the row pivoting, $satisfied\_q$ is set to true in line 23. Next, vector $w_i^{(i-1)}$ is computed in lines 26-30. In line 31, the pivot element $p_i^{(i-1)}$ is set equal to $q_i^{(i-1)}$. The key relation $p_j^{(i-1)} = (w_i^{(i-1)})^T(\Pi A \Sigma) e_j$, for $j \geq i+1$, is used to compute the vector $(p_{i+1}^{(i-1)}, \cdots, p_n^{(i-1)})$ in lines 32-34. After that, the column pivoting strategy is applied in lines 35-41 and matrix Σ is updated. Since the balance of the row pivoting has been ruined, then $satisfied\_q$ is set to false in line 37. After the column pivoting, $satisfied\_p$ should be set equal to true in line 42 and the algorithm will alternate between the row and the column pivoting until the desired pivot element will be computed. After the internal while loop of Algorithm 1, the i-th row and the i-th column of matrices L and U are computed and dropped in lines 45-48. After that, element $q_i^{(i-1)}$ is defined as the (i,i) entry of matrix D, i.e., d_{ii}. At the end of step n of Algorithm 1, the factorization $\Pi A \Sigma \approx M = LDU$ will be computed where $\Pi = \Pi_{n-1} \cdots \Pi_1$ and $\Sigma = \Sigma_1 \cdots \Sigma_{n-1}$. Matrix M is termed the left-looking version of RIF preconditioner with complete pivoting.

3. Numerical Results

In this section, we report the results of $GMRES(30)$ method to solve the original and the right preconditioned linear systems. Preconditioners are left-looking RIF with and without pivoting. In Table 1, notation $LLRIF$ is used for the left-looking version of RIF and $LLRIFP(1.0)$ indicates the left-looking version of RIF with pivoting

Algorithm 1

Input: Let $A \in \mathbb{R}^{n\times n}$, $U = L = I_n$, $\Pi = \Sigma = I_n$, $\tau_w, \tau_z, \tau_l, \tau_u \in (0,1)$ be drop tolerances and prescribe a tolerance $\alpha \in (0,1]$.

Output: $\Pi A \Sigma \approx LDU$.

for $i = 1$ to n **do**
  $m_i = n_i = 0$, $iter = 0$
  $satisfied\_p = satisfied\_q = false$
  while not $satisfied\_q$ **do**
    $z_i^{(0)} = e_i$
    $iter = iter + 1$
    for $j = 1$ to $i-1$ **do**
      $p_i^{(j-1)} = e_j^T(\Pi A \Sigma) z_i^{(j-1)}$
      $z_i^{(j)} = z_i^{(j-1)} - (\frac{p_i^{(j-1)}}{p_j^{(j-1)}}) z_j^{(j-1)}$
      for all $l \leq j$, if $|z_{li}^{(j)}| < \tau_z$, then set $z_{li}^{(j)} = 0$
    end for
    If $iter = 1$, then set $q_i^{(i-1)} = e_i^T(\Pi A \Sigma) z_i^{(i-1)}$. Otherwise set $q_i^{(i-1)} = p_i^{(i-1)}$
    for $j = i+1$ to n **do**
      $q_j^{(i-1)} = e_j^T(\Pi A \Sigma) z_i^{(i-1)}$
    end for
    if $|q_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |q_m^{(i-1)}|$ **then**
      $m_i = m_i + 1,\ \pi_{m_i}^{(i-1)} = I_n$.
      $satisfied\_p = false$
      choose k such that $|q_k^{(i-1)}| = max_{m \geq i+1} |q_m^{(i-1)}|$.
      Interchange rows i and k of $\pi_{m_i}^{(i-1)}$ and elements $q_i^{(i-1)}$ and $q_k^{(i-1)}$
      $\Pi = \pi_{m_i}^{(i-1)} \Pi$
    end if
    $satisfied\_q = true$
    if not $satisfied\_p$ **then**
      $w_i^{(0)} = e_i$
      for $j = 1$ to $i-1$ **do**
        $q_i^{(j-1)} = (w_i^{(j-1)})^T (\Pi A \Sigma) e_j$
        $w_i^{(j)} = w_i^{(j-1)} - (\frac{q_i^{(j-1)}}{q_j^{(j-1)}}) w_j^{(j-1)}$
        for all $l \leq j$, if $|w_{li}^{(j)}| < \tau_w$, then set $w_{li}^{(j)} = 0$
      end for
      $p_i^{(i-1)} = q_i^{(i-1)}$
      for $j = i+1$ to n **do**
        $p_j^{(i-1)} = (w_i^{(i-1)})^T (\Pi A \Sigma) e_j$
      end for
      if $|p_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |p_m^{(i-1)}|$ **then**
        $n_i = n_i + 1,\ \sigma_{n_i}^{(i-1)} = I_n$
        $satisfied\_q = false$
        choose l such that $|p_l^{(i-1)}| = \max_{m \geq i+1} |p_m^{(i-1)}|$
        Interchange columns i and l of $\sigma_{n_i}^{(i-1)}$ and elements $p_i^{(i-1)}$ and $p_l^{(i-1)}$
        $\Sigma = \Sigma \sigma_{n_i}^{(i-1)}$
      end if
      $satisfied\_p = true$
    end if
  end while
  for $j = 1$ to $i-1$ **do**
    $L_{ij} = \frac{q_i^{(j-1)}}{q_j^{(j-1)}}$, $U_{ji} = \frac{p_i^{(j-1)}}{p_j^{(j-1)}}$
    If $|L_{ij}| < \tau_l$, then set $L_{ij} = 0$. Also if $|U_{ji}| < \tau_u$, then set $U_{ji} = 0$.
  end for
  $d_{ii} = q_i^{(i-1)}$
end for
Return $L = (L_{ij})_{1 \leq i,j \leq n}$, $U = (U_{ij})_{1 \leq i,j \leq n}$, $D = diag(d_{ii})_{1 \leq i \leq n}$, Π and Σ.

TABLE 1.

	properties		*No preconditioning*		*LLRIFP*(1.0)			*LLRIF*		
	n	*nnz*	*it*	*Itime*	*density*	*it*	*Ttime*	*density*	*it*	*Ttime*
fpga_dcop_12	1220	5892	+	+	1.842	45	0.078	0.9628	79	0.062
raefsky6	3402	137845	1353	5.72	0.995	5	0.484	0.274	7	0.406
sherman4	1104	3786	558	0.36	1.802	26	0.031	1.239	46	0.031
fpga_dcop_14	1220	5892	+	+	1.813	77	0.093	0.954	1144	1.296
epb3	84617	463625	+	+	1.560	207	20.43	1.005	316	24.312
poisson3Da	13514	352762	261	3.50	2.863	60	5.390	0.337	130	3.765

that uses $\alpha = 1.0$ for the row and the column pivoting. We have considered 6 linear systems with the coefficient matrices from reference [2]. The right hand side vector of these systems is Ae where $e = [1, \cdots, 1]^T$. The code of left-looking version of RIF with pivoting is written in $Fortran$ 77 and the codes of $GMRES$ and left-looking version of RIF are downloaded from the $Sparskit$ [4] and $Sparslab$ [1] packages. In all the tests, parameters τ_w, τ_z, τ_l and τ_u are set equal to 0.1. In Table 1, n is the dimension and nnz is the number of nonzero entries of the matrix. In columns 4 and 5 of this table, it and $Itime$ are the number of iterations and the iteration time of $GMRES(30)$ method with no preconditioning. In columns 7 and 10 and in columns 8 and 11 of this table, it is the number of iterations and $Ttime$ is the total time of $GMRES(30)$ method that solves the right preconditioned linear systems. For all the tests of this table, the convergence criterion is satisfied when the relative residual is less than 10^{-8}. A + sign indicates that this criterion has not been satisfied in 5000 number of iterations. The parameter $density$ in this table is defined as $density = \frac{nnz(L)+nnz(U)}{nnz(A)}$ where $nnz(L)$ and $nnz(U)$ are the number of nonzero entries of L and U factors. Numerical experiments indicate that for all matrices, $LLRIFP(1.0)$ is denser than $LLRIF$. The results also indicate that $LLRIFP(1.0)$ makes the $GMRES(30)$ method converge in less number of iterations than $LLRIF$.

REFERENCES

1. M. Benzi and M. Tůma, *Sparslab software package.* http://www2.cs.cas.cz/ tuma/sparslab.html. Accessed 2013.
2. T. Davis, *University of Florida Sparse Matrix Collection*, http://www.cise.ufl.edu/research/sparse/matrices, accessed 2013.
3. Y. Saad, *Iterative Methods for Sparse Linear Systems.* PWS publishing, New York., 1996.
4. Y. Saad, *Sparskit and sparse examples.* http://www-users.cs.umn.edu/ saad/software. Accessed 2013.