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Right-looking version of robust incomplete factorization preconditioner with pivoting

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Abstract

In this paper, we use a complete pivoting strategy for the right-looking version of Robust Incomplete Factorization preconditioner and study effect of this pivoting.
Keywords: ILU factorization, right-looking version of RIF preconditioner, Krylov subspace methods.

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1 Introduction

Consider the linear system of equations of the form

$$Ax = b, \tag{1}$$

where the coefficient matrix $A \in \mathbb{R}^{n \times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. Krylov subspace methods can be used to solve this system [5]. An implicit preconditioner M for system (1) is an approximation of matrix A, *i.e.*, $M \approx A$. If M is a good approximation of A, then it can be used as the right preconditioner for system (1). In this case, instead of solving system (1) it is better to solve the right preconditioned system $AM^{-1}u = b$ where $M^{-1}u = x$ by the Krylov subspace methods.

Suppose that there is the factorization $A = LDU$ for matrix A; where L and U^T are unit lower triangular matrices and D is a diagonal matrix. Also suppose that dropping be applied on L, U and D. Then, matrix M which is $A \approx M = LDU$ is an implicit preconditioner for system (1). This preconditioner is also termed as an ILU preconditioner.

In this paper, we present a complete pivoting strategy for right-looking version of RIF preconditioner. To test effectiveness of such a pivoting, we have generated several linear systems. The coefficient matrices are taken from University of Florida sparse matrix collection [4]. Then we have computed the right-looking version of RIF with pivoting for such systems and have solved the right preconditioned linear systems by the $GMRES(30)$ Krylov subspace method [5].

2 Right-looking version of RIF preconditioner with complete pivoting

An explicit preconditioner M for system (1) is an approximation of matrix A^{-1}, *i.e.*, $M \approx A^{-1}$. The most well-known explicit preconditioner is the $AINV$ preconditioner [1]. This preconditioner has three factors in the form $A^{-1} \approx M = ZD^{-1}W^T$ where Z and W are unit upper triangular matrices and D is a diagonal matrix. There are two left and right-looking versions for this preconditioner. In [3], Bollhoefer and Saad could present a complete pivoting strategy for the right-looking version of $AINV$ preconditioner.

In 2003, Benzi and Tůma computed an incomplete factorization of a symmetric positive definite matrix A in the form of $A \approx LDL^T$ as by-product of the $AINV$ preconditioner. This preconditioner is termed Robust Incomplete Factorization or RIF [2]. There are also two left and right-looking versions for this preconditioner. In this paper, we focus on the nonsymmetric version of this preconditioner. The L and U factors of this preconditioner are computed independently. Algorithm 1, computes the right-looking version of this preconditioner and uses the complete pivoting strategy. At step i of Algorithm 1, i-th column of matrix L and i-th row of matrix U are computed, independently. More precisely, matrix L is computed column-wise and matrix U is computed row-wise. At the end of this algorithm, factorization $\Pi A \Sigma \approx LDU$ is computed where Π and Σ are the row and column permutation matrices, respectively.

3 Numerical Results

In this section, we report results of $GMRES(30)$ method to solve the original and the right preconditioned linear systems. Preconditioners are right-looking version of RIF with and without pivoting. We have used the notation $RLRIF$ to indicate the right-looking version of RIF preconditioner in Tables 2 and 3. We have also considered the notation $RLRIFP(1.0)$ in these tables to indicate the right-looking version of RIF with pivoting that uses parameter $\alpha = 1.0$ for column and row pivoting. We have generated 8 artificial linear systems $Ax = b$ with the nonsymmetric coefficient matrices from the University of Florida Sparse Matrix Collection [4]. Vector b is Ae in which $e = [1, ..., 1]^T$. We have written code of right-looking version of RIF with pivoting in Fortran 90. We have selected τ_l, τ_u, τ_w and τ_z equal to 0.1.

In Table 1, n is the dimension and nnz is the number of nonzero entries of the matrix. it is the number of iterations of $GMRES(30)$ method with no preconditioning and $Itime$ is its iteration time. This parameter is in seconds. A + means that there is no convergence in 5000 number of iterations. In Tables 1 and 3, convergence criterion is satisfied when the relative residual is less than 10^{-10}.

In Table 2, $Ptime$ is the preconditioning time and $density = \frac{nnz(L)+nnz(U)}{nnz(A)}$. $Ptime$ is in seconds. For all matrices, the density of $RLRIFP(1.0)$ and $RLRIF$ preconditioners are so close to each other. This gives a proper atmosphere to compare number of iterations of $GMRES(30)$, when these two preconditioners are used as the right preconditioner. For matrix $orsirr\_2$, $Ptime$ of $RLRIFP(1.0)$ preconditioner is smaller than $Ptime$ of $RLRIF$ preconditioner. But for all other matrices, $Ptime$ of $RLRIFP(1.0)$ preconditioner is greater than or equal to $Ptime$ of $RLRIF$ preconditioner.

Algorithm 1 (Right-looking version of RIF with complete pivoting)

Input: $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $\tau_w, \tau_z \in (0,1)$ be drop tolerances and prescribe a tolerance $\alpha \in (0,1]$.
Output: $\Pi A \Sigma \approx LDU$

$\Pi = \Sigma = I_n$.
$w_i^{(0)} = e_i, \quad z_i^{(0)} = e_i, \quad 1 \le i \le n.$
for $i = 1$ to n **do**
  $m_i = n_i = 0$
  $satisfied\_p = false, \quad satisfied\_q = false.$
  while not $satisfied\_p$ **do**
    for $j = i$ to n **do**
      $p_j^{(i-1)} = (w_j^{(i-1)})^T (\Pi A \Sigma) z_i^{(i-1)}$.
    end for
    if $|p_i^{(i-1)}| < \alpha\ max_{m\ge i} |p_m^{(i-1)}|$ **then**
      $m_i = m_i + 1, \quad \pi_{m_i}^{(i-1)} = I_n$
      $satisfied\_q = false$, choose k such that $|p_k^{(i-1)}| = max_{m\ge i}|p_m^{(i-1)}|$.
      Interchange columns i and k of $W - I$ and rows i and k of $L - I$ and $\pi_{m_i}^{(i-1)}$ and elements $p_i^{(i-1)}$ and $p_k^{(i-1)}$.
      $\Pi = \pi_{m_i}^{(i-1)} \Pi$
    end if
    $satisfied\_p = true$
    for $j = i$ to n **do**
      $q_j^{(i-1)} = (z_j^{(i-1)})^T (\Pi A \Sigma)^T w_i^{(i-1)}$.
    end for
    if not $satisfied\_q$ **then**
      if $|q_i^{(i-1)}| < \alpha\ max_{m\ge i}|q_m^{(i-1)}|$ **then**
        $n_i = n_i + 1, \quad \sigma_{n_i}^{(i-1)} = I_n$
        $satisfied\_p = false$, choose l such that $|q_l^{(i-1)}| = max_{m\ge i}|q_m^{(i-1)}|$.
        Interchange columns i and l of $Z - I$, $U - I$ and $\sigma_{n_i}^{(i-1)}$ and elements $q_i^{(i-1)}$ and $q_l^{(i-1)}$.
        $\Sigma = \Sigma \sigma_{n_i}^{(i-1)}$
      end if
    end if
    $satisfied\_q$=true.
  end while
  $d_{ii} = p_i^{(i-1)}$.
  for $j = i + 1$ to n **do**
    $p_j^{(i-1)} = \frac{p_j^{(i-1)}}{d_{ii}}, \quad q_j^{(i-1)} = \frac{q_j^{(i-1)}}{d_{ii}}$.
    $L_{ji} = p_j^{(i-1)}, \quad U_{ij} = q_j^{(i-1)}$.
    apply dropping rule to L_{ji} and to U_{ij} if their absolute values are less than τ_l and τ_u.
    $w_j^{(i)} = w_j^{(i-1)} - p_j^{(i-1)} w_i^{(i-1)}, \quad z_j^{(i)} = z_j^{(i-1)} - q_j^{(i-1)} z_i^{(i-1)}$.
    for all $l \le i$ apply a dropping rule to $w_{lj}^{(i)}$ and to $z_{lj}^{(i)}$, if their absolute values are less than τ_w and τ_z.
  end for
end for
Return $L = (L_{ij})_{1\le i,j\le n}$, $U = (U_{ij})_{1\le i,j\le n}$, $D = diag(d_{ii})_{1\le i\le n}$, Π and Σ

In Table 3, results of $GMRES(30)$ method are presented. In this table, it is again the number of iterations of this Krylov subspace method and $Ttime$ is its total time which is defined as the iteration time plus the preconditioning time. Comparing columns 2 and 4 of this table, indicates that except for matrix $raefsky6$, the number of iterations of $GMRES(30)$, when one uses the $RLRIFP(1.0)$ preconditioner, is less than or equal to the number of iterations of this Krylov subspace method when one applies the $RLRIF$ preconditioner. Columns 3 and 5 of this table, indicate that for 6 matrices $Ttime$ of $RLRIFP(1.0)$ preconditioner is less than $Ttime$ of $RLRIF$ preconditioner and for 2 other matrices this is vice versa. Comparing column 2 of Table 3 with column 4 of Table 1, shows that $RLRIFP(1.0)$ preconditioner is useful to decrease the number of iterations of $GMRES(30)$ method.

4 Conclusion

In this paper, we presented right-looking version of RIF preconditioner with pivoting. We used parameter $\alpha = 1.0$ to compute this preconditioner. We also used this preconditioner as the right preconditioner to solve several linear systems. Numerical experiments indicate the fact that, although $Ptime$ of right-looking version of RIF preconditioner with pivoting for most of the tests is greater than $Ptime$ of right-looking version of RIF preconditioner, but the right-looking version of RIF with pivoting is more effective than right-looking version of RIF to decrease the number of iterations of $GMRES(30)$ method.

Table 1: matrix properties and results of $GMRES(30)$ with no preconditioning.

Matrix	n	nnz	It	$Itime$
$add20$	2395	17319	1022	0.7500
$raefsky1$	3242	294276	+	+
$raefsky2$	3242	294276	+	+
$raefsky5$	6316	168658	+	+
$raefsky6$	3402	137845	+	+
$sherman4$	1104	3786	722	0.2188
$orsirr\_1$	1030	6858	+	+
$orsirr\_2$	886	5970	+	+

Table 2: properties of $RLRIFP(1.0)$ and $RLRIF$ preconditioners.

method	$RLRIFP(1.0)$		$RLRIF$	
	$Ptime$	$density$	$Ptime$	$density$
$add20$	0.0468	0.5556	0.0312	0.5354
$raefsky1$	0.2187	0.0895	0.0781	0.0911
$raefsky2$	0.3125	0.1458	0.1094	0.1473
$raefsky5$	0.1093	0.2526	0.0625	0.2602
$raefsky6$	0.0781	0.0872	0.0781	0.2739
$sherman4$	0.0156	1.2556	0.0156	1.2361
$orsirr\_1$	0.0312	0.6211	0.0156	0.5426
$orsirr\_2$	0.0312	0.6229	0.0313	0.5457

Table 3: results of $GMRES(30)$ using $RLRIFP(1.0)$ and $RLRIF$ preconditioners

method	$RLRIFP(1.0)$		$RLRIF$	
	it	$Ttime$	it	$Ttime$
$add20$	15	0.0468	21	0.0469
$raefsky1$	642	2.4218	814	2.8281
$raefsky2$	636	2.5937	738	2.7188
$raefsky5$	10	0.1250	10	0.0938
$raefsky6$	10	0.0937	9	0.0938
$sherman4$	61	0.0625	69	0.0313
$orsirr\_1$	49	0.0468	106	0.0469
$orsirr\_2$	49	0.0312	105	0.0625

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