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COMPLETE PIVOTING STRATEGY FOR THE LEFT-LOOKING VERSION OF $AINV$ PRECONDITIONER

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Abstract. In this paper, we use a complete pivoting strategy for the left-looking version of $AINV$ preconditioner.

1. Introduction

Consider the linear system of equations of the form $Ax = b$, where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. Krylov subspace methods can be used to solve this system [4]. An explicit preconditioner M for this system is an approximation of matrix A^{-1}, *i.e.*, $M \approx A^{-1}$. If the explicit preconditioner M is a good approximation of A^{-1}, then it is better to solve the right preconditioned system $AMu = b$ where $Mu = x$ by the Krylov subspace methods. The most well-known explicit preconditioner is the $AINV$ preconditioner [1]. This preconditioner has three factors in the form $A^{-1} \approx M = ZD^{-1}W^T$ where Z and W are unit upper triangular matrices and D is a diagonal matrix. There are two left and right-looking versions for this preconditioner. In this paper, we extend pivoting for the left-looking version of this preconditioner.

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2. Left-looking version of $AINV$ preconditioner with complete pivoting

Algorithm 1, computes the left-looking version of $AINV$ preconditioner with pivoting. Suppose that no dropping be applied in this algorithm. Also suppose that Π_k and Σ_k for $1 \leq k \leq i-1$ are the computed row and column permutation matrices at steps 1 to $i-1$ of this algorithm. At the beginning of step i of this algorithm, at first, lines 2-4 are set and then, vector $z_i^{(i-1)}$ is computed in lines 7-10. The pivot element $q_i^{(i-1)}$ is computed in line 11. In lines 12-14, the relation $q_j^{(i-1)} = e_j^T(\Pi A \Sigma) z_i^{(i-1)}$, for $j \geq i+1$, gives the chance to compute the vector $(q_{i+1}^{(i-1)}, \cdots, q_n^{(i-1)})$. After that, row pivoting strategy is done in lines 15-21 and matrix Π is updated. After the row pivoting, $satisfied\_q$ is set equal to true in line 22. Next, vector $w_i^{(i-1)}$ is computed in lines 24-27. In line 28, the pivot element $p_i^{(i-1)}$ is set equal to $q_i^{(i-1)}$. In lines 29-31, the key relation $p_j^{(i-1)} = (w_i^{(i-1)})^T(\Pi A \Sigma) e_j$, for $j \geq i+1$, is used to compute the vector $(p_{i+1}^{(i-1)}, \cdots, p_n^{(i-1)})$. After that, column pivoting strategy is applied in lines 32-38 and matrix Σ is updated. After the column pivoting, $satisfied\_p$ is set equal to true in line 39. After the internal while loop of Algorithm 1, at first, we drop entries of the computed columns $z_i^{(i-1)}$ and $w_i^{(i-1)}$ in line 42. Then, element $q_i^{(i-1)}$ is defined as the (i,i) entry of matrix D, i.e., d_{ii}. At the end of step n of this algorithm; the factorization $(\Pi A \Sigma)^{-1} \approx M = ZD^{-1}W^T$ will be computed where $\Pi = \Pi_{n-1} \cdots \Pi_1$ and $\Sigma = \Sigma_1 \cdots \Sigma_{n-1}$. In this case, matrix M is termed the left-looking version of $AINV$ preconditioner with complete pivoting.

3. Numerical Results

In this section, we report results of $GMRES(16)$ method to solve the original and the right preconditioned linear systems. Preconditioners are left-looking $AINV$ with and without pivoting. In the table, notation $LLAINV$ is used for the left-looking version of $AINV$ and $LLAINVP(1.0)$ indicates the left-looking version of $AINV$ with pivoting which is computed by using $\alpha = 1.0$ for row and column pivoting. We have considered 6 linear systems with the coefficient matrices from reference [3]. The right hand side vector of these systems is Ae where $e = [1, \cdots, 1]^T$. We have written code of left-looking version of $AINV$

Algorithm 1 (**Left-looking** $AINV$ **preconditioner with complete pivoting**)

Input: Let $A = \in \mathbb{R}^{n\times n}$, $\Pi = \Sigma = I_n$, $\tau_w, \tau_z \in (0,1)$ be drop tolerances and prescribe a tolerance $\alpha \in (0,1]$.

Output: $(\Pi A \Sigma)^{-1} \approx Z D^{-1} W^T$.

for $i = 1$ to n **do**
  $w_i^{(0)} = z_i^{(0)} = e_i$
  $m_i = n_i = 0,\ iter = 0$
  $satisfied\_p = satisfied\_q = false$
  while not $satisfied\_q$ **do**
    $iter = iter + 1$
    for $j = 1$ to $i-1$ **do**
      $p_i^{(j-1)} = e_j^T (\Pi A \Sigma) z_i^{(j-1)}$
      $z_i^{(j)} = z_i^{(j-1)} - (\frac{p_i^{(j-1)}}{p_j^{(j-1)}}) z_j^{(j-1)}$
    end for
    If $iter = 1$, then set $q_i^{(i-1)} = e_i^T (\Pi A \Sigma) z_i^{(i-1)}$. Otherwise set $q_i^{(i-1)} = p_i^{(i-1)}$
    for $j = i+1$ to n **do**
      $q_j^{(i-1)} = e_j^T (\Pi A \Sigma) z_i^{(i-1)}$
    end for
    if $|q_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |q_m^{(i-1)}|$ **then**
      $m_i = m_i + 1,\ \pi_{m_i}^{(i-1)} = I_n$
      $satisfied\_p = false$
      choose k such that $|q_k^{(i-1)}| = max_{m \geq i+1} |q_m^{(i-1)}|$
      Interchange rows i and k of $\pi_{m_i}^{(i-1)}$ and elements $q_i^{(i-1)}$ and $q_k^{(i-1)}$
      $\Pi = \pi_{m_i}^{(i-1)} \Pi$
    end if
    $satisfied\_q = true$
    if not $satisfied\_p$ **then**
      for $j = 1$ to $i-1$ **do**
        $q_i^{(j-1)} = (w_i^{(j-1)})^T (\Pi A \Sigma) e_j$
        $w_i^{(j)} = w_i^{(j-1)} - (\frac{q_i^{(j-1)}}{q_j^{(j-1)}}) w_j^{(j-1)}$
      end for
      $p_i^{(i-1)} = q_i^{(i-1)}$
      for $j = i+1$ to n **do**
        $p_j^{(i-1)} = (w_i^{(i-1)})^T (\Pi A \Sigma) e_j$
      end for
      if $|p_i^{(i-1)}| < \alpha\ max_{m \geq i+1} |p_m^{(i-1)}|$ **then**
        $n_i = n_i + 1,\ \sigma_{n_i}^{(i-1)} = I_n$
        $satisfied\_q = false$
        choose l such that $|p_l^{(i-1)}| = max_{m \geq i+1} |p_m^{(i-1)}|$
        Interchange columns i and l of $\sigma_{n_i}^{(i-1)}$ and elements $p_i^{(i-1)}$ and $p_l^{(i-1)}$
        $\Sigma = \Sigma \sigma_{n_i}^{(i-1)}$
      end if
      $satisfied\_p = true$
    end if
  end while
  For all $l \leq i$, if $|z_{li}^{(i-1)}| < \tau_z$, then set $z_{li}^{(i-1)} = 0$. Also if $|w_{li}^{(i-1)}| < \tau_w$, then set $w_{li}^{(i-1)} = 0$
  $d_{ii} = q_i^{(i-1)}$
end for
Return $Z = [z_1^{(0)}, z_2^{(1)}, \cdots, z_n^{(n-1)}]$, $W = [w_1^{(0)}, w_2^{(1)}, \cdots, w_n^{(n-1)}]$, $D = diag(d_{ii})_{1 \leq i \leq n}$, Π and Σ.

method	*properties*		*no precond*		$LLAINVP(1.0)$				$LLAINV$			
	n	nnz	It	$Itime$	$Ptime$	$density$	It	$Ttime$	$Ptime$	$density$	It	$Ttime$
$raefsky5$	6316	168658	70	0.343	1.28	0.235	7	1.32	0.406	0.512	9	0.453
$orsirr\_1$	1030	6858	+	+	0.25	0.870	66	0.281	0.156	0.644	+	+
$orsirr\_2$	886	5970	+	+	0.187	0.906	65	0.218	0.125	0.652	+	+
$cdde1$	961	4681	2682	1.015	0.453	1.787	85	0.1562	0.046	1.66	1396	0.75
$raefsky6$	3402	137845	1819	6.375	4.312	0.090	7	4.34	1.98	0.257	11	2.03
$orsreg\_1$	2205	14133	727	0.687	5.546	0.901	66	5.625	3.15	0.667	+	+

with pivoting in Fortran 77. Codes of left-looking version of $AINV$ and $GMRES$ are taken from the $Sparslab$ [2] and $Sparskit$ [5] packages, respectively. In all the tests, parameters τ_w and τ_z are set equal to 0.1. In the table, n is the dimension and nnz is the number of nonzero entries of the matrix. In columns 4 and 5, It is the number of iterations of $GMRES(16)$ method with no preconditioning and $Itime$ is its iteration time. This parameter is in second. In columns 8 and 12 of the table, It is the number of iterations of $GMRES(16)$ that solves the right preconditioned linear systems. In columns 9 and 13, $Ttime$ is the total time to solve the right preconditioned linear systems. $Ttime$ is the iteration time plus the preconditioning time and is also in second. For all the tests, the convergence criterion is satisfied when the relative residual is less than 10^{-8}. In the table, a + sign means that the convergence criterion is not satisfied in 2500 number of iterations. In the table, $Ptime$ is the preconditioning time which is also in second and $density = \frac{nnz(Z)+nnz(W)}{nnz(A)}$ where $nnz(Z)$ and $nnz(W)$ are the number of nonzero entries of Z and W factors. For all matrices, $Ptime$ of $LLAINVP(1.0)$ is greater than $Ptime$ of $LLAINVP(1.0)$ but $LLAINVP(1.0)$ is more effective than $LLAINV$ to reduce the number of iterations of $GMRES(16)$ method.

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