

## Complete pivoting strategy to compute the IULBF preconditioner

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### Abstract

In this paper, a complete pivoting strategy to compute the IULBF preconditioner is presented.

**Keywords:** pivoting, IULBF preconditioner.

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## 1 Introduction

Consider the linear system of equations of the form  $Ax = b$ , where the coefficient matrix  $A \in \mathbb{R}^{n \times n}$  is nonsingular, large, sparse and nonsymmetric and also  $x, b \in \mathbb{R}^n$ . An *IUL* preconditioner  $M$  for this system is in the form of  $M = UDL \approx A$ . This preconditioner will change the original system to the left preconditioned system  $M^{-1}Ax = M^{-1}b$ . For a proper preconditioner, instead of solving the original system, it is better to solve the left preconditioned system by the Krylov subspace methods [4]. In [1, 2], we have proposed an *IUL* preconditioner for system  $Ax = b$ . This preconditioner is termed the *IULBF*.

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### Algorithm 1 ( IULBF preconditioner )

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**Input:**  $A \in \mathbb{R}^{n \times n}$  and  $\tau_z, \tau_w, \tau_l, \tau_u \in (0, 1)$  be drop tolerances parameters.

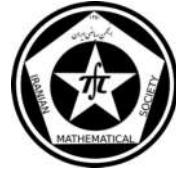
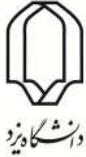
**Output:**  $A \approx UDL$

1. **for**  $i = n$  to  $1$  **do**
  2.    $w_i^{(0)} = e_i^T$ ,    $z_i^{(0)} = e_i$ .
  3.   **for**  $j = i + 1$  to  $n$  **do**
  4.      $p_j^{(i-1)} = e_i^T A z_j^{(n-j)}$     $q_j^{(i-1)} = w_j^{(n-j)} A e_i$
  5.      $U_{ij} = \frac{p_j^{(i-1)}}{d_{jj}}$ ,    $L_{ji} = \frac{q_j^{(i-1)}}{d_{jj}}$
  6.     If  $|L_{ji}| < \tau_l$ , then set  $L_{ji} = 0$ . Also if  $|U_{ij}| < \tau_u$ , then set  $U_{ij} = 0$
  7.      $z_i^{(j-i)} = z_i^{(j-i-1)} - \frac{q_j^{(i-1)}}{d_{jj}} z_j^{(n-j)}$ ,    $w_i^{(j-i)} = w_i^{(j-i-1)} - \frac{p_j^{(i-1)}}{d_{jj}} w_j^{(n-j)}$
  8.     For all  $l \geq j$ , if  $|z_{li}^{(j-i)}| < \tau_z$  and  $|w_{il}^{(j-i)}| < \tau_w$ , then set  $z_{li}^{(j-i)} = 0$  and  $w_{il}^{(j-i)} = 0$
  9.   **end for**
  10.    $d_{ii} = w_i^{(n-i)} A e_i$
  11. **end for**
  12. Return  $U = (U_{ij})_{1 \leq i,j \leq n}$ ,  $D = \text{diag}(d_{ii})_{1 \leq i \leq n}$  and  $L = (L_{ji})_{1 \leq j,i \leq n}$ .
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Algorithm 1, computes the *IULBF* preconditioner. In this algorithm, matrices  $L$  and  $U$  are computed column-wise and row wise, respectively.

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## 2 Pivoting strategy for the IULBF preconditioner

Algorithm 2, computes the IULBF preconditioner which is coupled with complete pivoting strategy. The pivoting strategy of this algorithm is based on the complete pivoting strategy of the Backward IJK version of Gaussian Elimination process. In lines 16 and 35 of this algorithm we use the parameter  $\alpha \in (0, 1]$  to control the pivoting process.

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**Algorithm 2 (IULBF preconditioner coupled with complete pivoting strategy)**

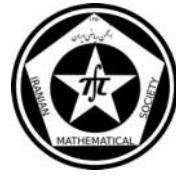
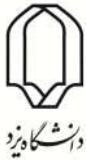
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**Input:** Let  $A \in \mathbb{R}^{n \times n}$ ,  $U = L = \Pi = \Sigma = I_n$ ,  $\tau_z, \tau_w, \tau_l, \tau_u \in (0, 1)$  be drop tolerances and prescribe a pivoting tolerance  $\alpha \in (0, 1]$ .

**Output:**  $\Pi A \Sigma \approx U D L$ .

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1. for  $i = n$  to  $1$  do
2.    $m_i = n_i = iter = 0$ 
3.   satisfied_p = satisfied_q = false
4.   while not satisfied_p do
5.     iter = iter + 1
6.      $z_i^{(0)} = e_i$ 
7.     for  $j = i + 1$  to  $n$  do
8.        $q_j^{(i-1)} = w_j^{(n-j)} (\Pi A \Sigma) e_i$ 
9.        $z_i^{(j-i)} = z_i^{(j-i-1)} - (\frac{q_j^{(i-1)}}{d_{jj}}) z_j^{(n-j)}$ 
10.      For all  $l \geq j$ , if  $|z_l^{(j-i)}| < \tau_z$ , then set  $z_l^{(j-i)} = 0$ .
11.    end for
12.    If  $iter = 1$ , then set  $p_i^{(i-1)} = e_i^T (\Pi A \Sigma) z_i^{(n-i)}$ . Otherwise set  $p_i^{(i-1)} = q_i^{(i-1)}$ 
13.    for  $j = i - 1$  to  $1$  do
14.       $p_i^{(j-1)} = e_j^T (\Pi A \Sigma) z_i^{(n-i)}$ 
15.    end for
16.    if  $|p_i^{(i-1)}| < \alpha \max_{m \leq i} |p_i^{(m-1)}|$  then
17.       $m_i = m_i + 1$ ,  $\pi_{m_i}^{(i)} = I_n$ .
18.      satisfied_q = false
19.      Choose  $k$  such that  $|p_i^{(k-1)}| = \max_{m \leq i} |p_i^{(m-1)}|$ .
20.      interchange the rows  $i$  and  $k$  of  $\pi_{m_i}^{(i)}$  and the elements  $p_i^{(i-1)}$  and  $p_i^{(k-1)}$ 
21.       $\Pi = \pi_{m_i}^{(i)} \Pi$ 
22.    end if
23.    satisfied_p = true
24.    if not satisfied_q then
25.       $w_i^{(0)} = e_i^T$ 
26.      for  $j = i + 1$  to  $n$  do
27.         $p_j^{(i-1)} = e_i^T (\Pi A \Sigma) z_j^{(n-j)}$ 
28.         $w_i^{(j-i)} = w_i^{(j-i-1)} - (\frac{p_j^{(i-1)}}{d_{jj}}) w_j^{(n-j)}$ 
29.        For all  $l \geq j$ , if  $|w_l^{(j-i)}| < \tau_w$ , then set  $w_l^{(j-i)} = 0$ .
30.      end for
31.       $q_i^{(i-1)} = p_i^{(i-1)}$ 
32.      for  $j = i - 1$  to  $1$  do
33.         $q_i^{(j-1)} = w_i^{(n-i)} (\Pi A \Sigma) e_j$ 
34.      end for
35.      if  $|q_i^{(i-1)}| < \alpha \max_{m \leq i} |q_i^{(m-1)}|$  then
36.         $n_i = n_i + 1$ ,  $\sigma_{n_i}^{(i)} = I_n$ 
37.        satisfied_p = false
38.        Choose  $l$  such that  $|q_i^{(l-1)}| = \max_{m \leq i} |q_i^{(m-1)}|$ .
39.        interchange the columns  $i$  and  $l$  of  $\sigma_{n_i}^{(i)}$  and the elements  $q_i^{(i-1)}$  and  $q_i^{(l-1)}$ 
40.         $\Sigma = \Sigma \sigma_{n_i}^{(i)}$ 
41.      end if
42.      satisfied_q = true
43.    end if
44.  end while
45.   $d_{ii} = p_i^{(i-1)}$ 
46.  for  $j = i + 1$  to  $n$  do
47.     $L_{ji} = \frac{q_j^{(i-1)}}{d_{jj}}$ ,  $U_{ij} = \frac{p_j^{(i-1)}}{d_{jj}}$ 
48.    If  $|L_{ji}| < \tau_l$ , then set  $L_{ji} = 0$ . Also if  $|U_{ij}| < \tau_u$ , then set  $U_{ij} = 0$ .
49.  end for
50. end for
51. Return  $L = (L_{ji})_{1 \leq j, i \leq n}$ ,  $D = \text{diag}(d_{ii})_{1 \leq i \leq n}$ ,  $U = (U_{ij})_{1 \leq i, j \leq n}$ ,  $\Pi$  and  $\Sigma$ .
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### 3 Numerical results

In this section, we have considered 8 artificial linear systems where the coefficient matrices are downloaded from [3] and the exact solution of these systems is the vector  $[1, \dots, 1]^T$ . We have used two parameters 0.75 and 1.0 as  $\alpha$  to compute the *IULBF* preconditioner with complete pivoting strategy. We have used the command *GMRES* in Matlab software to solve the original and the left preconditioner systems. We have used 10 as the number of restarts for the *GMRES* method. The stopping criterion for all linear systems is satisfied when the relative residual is less than  $10^{-6}$ . We have considered the zero vector as the initial solution for all linear systems. The density of all preconditioners is defined as:

$$\text{density} = \frac{\text{nnz}(L) + \text{nnz}(U)}{\text{nnz}(A)},$$

where  $\text{nnz}(L)$ ,  $\text{nnz}(U)$  and  $\text{nnz}(A)$  refer to the number of nonzero entries of matrices  $L$ ,  $U$  and  $A$ , respectively. To compute all of the preconditioners we have considered all of the drop tolerance parameters equal to 0.1 .

Table 1, shows the matrix properties and the information of *GMRES* method to solve the original linear systems. In this table,  $n$  and  $\text{nnz}$  are the dimension and the number of nonzero entries of the matrix.

Table 1: matrix properties and information of the *GMRES*(10) method

Matrix	$n$	$\text{nnz}$	without preconditioner			
			outer	inner	flag	Itime
<i>bawa62</i>	62	450	161	2	0	0.5252
<i>tub100</i>	100	396	724	10	1	12.5472
<i>bwm200</i>	200	796	5000	10	1	14.3536
<i>saylr1</i>	238	1128	5000	10	1	13.2154
<i>cage7</i>	340	4380	2	8	0	0.0134
<i>tols340</i>	340	2196	3881	10	1	13.5088
<i>bfbw398</i>	398	1654	3	9	0	0.0778
<i>olm500</i>	500	1996	4023	10	1	9.0694

In all the tables, the parameters *outer*, *inner* and *flag* indicate the *outer* iterations, the *inner* iterations and the status of the convergence for *GMRES*(10) method.

Table 2: properties of the IULBF preconditioner

Method	IULBF				
	Matrix	density	outer	inner	flag
	<i>bawa62</i>	0.9111	2	8	0
	<i>tub100</i>	1.0051	1	10	0
	<i>bwm200</i>	1	4	4	0
	<i>saylr1</i>	0.9592	4	7	0
	<i>cage7</i>	0.4841	1	8	0
	<i>tols340</i>	0.9039	2	10	0
	<i>bfbw398</i>	0.8368	1	6	0
	<i>olm500</i>	1.1839	4	7	0

In Tables 1 – 3, when *flag* is equal to 0, it means that the method has been converged to the desired tolerance within the 2500 outer iterations. *flag* = 1 shows that we can not obtain the convergence in 2500 number of iterations.

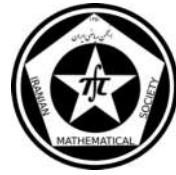
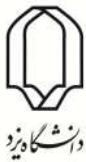


Table 3: properties of the IULBFP(0.75) and IULBFP(1.0) preconditioners

Method	IULBFP(0.75)						IULBFP(1.0))					
	Matrix	density	Rpiv	Cpiv	outer	inner	flag	density	Rpiv	Cpiv	outer	inner
bawa62	0.9022	3	2	2	8	0	0.9000	4	4	2	8	0
tub100	1.0657	24	22	1	10	0	1.1086	23	23	1	9	0
bwm200	1.1131	51	45	12	9	0	1.1256	51	51	10	2	0
sayrl1	0.9592	0	0	4	7	0	0.9592	0	0	4	7	0
cage7	0.4780	0	0	1	8	0	0.4780	0	0	1	8	0
tol340	0.3679	37	76	1	7	0	0.3657	40	77	1	7	0
bfbw398	0.8368	0	0	1	6	0	0.8368	0	0	1	6	0
olm500	0.9965	499	249	3	6	0	0.9965	499	249	3	6	0

In Tabe 3, notation *IULBFP*( $\alpha$ ) refers to the *IULBF* preconditioner with complete pivoting strategy which is computed by the parameter  $\alpha$ . The columns *Rpiv* and *Cpiv* show the total number of row and column pivoting. In Tables 2 and 3, the information in the columns *flag*, *outer* and *inner* associated to the three preconditioners indicate that for all of the matrices, one of the preconditioners *IULBFP*(1.0) or *IULBFP*(0.75) gives better results of the *GMRES*(10) method than the *IULBF* preconditioner. This means that the complete pivoting strategy with one of the values  $\alpha = 1.0$  or  $\alpha = 0.75$  has a good effect on the quality of the *IULBF* preconditioner.

If we compare the columns *flag*, *outer* and *inner* in Table 2 by the columns *flag*, *outer* and *inner* of Table 1, then it is clear that the two preconditioners *IULBFP*(1.0) and *IULBFP*(0.75) are useful tools to decrease the number of iterations of the *GMRES*(10) method.

## References

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