

## Pivoting strategy for an $ILU$ preconditioner

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### Abstract

In this paper, a complete pivoting strategy for the right-looking version of  $RIF - NS$  preconditioner is presented.

**Keywords:** preconditioning, pivoting, right-looking version of  $RIF - NS$  preconditioner

**Mathematics Subject Classification [2010]:** 65F10, 65F50, 65F08.

## 1 Introduction

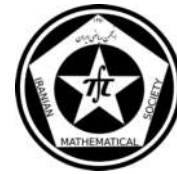
Consider the linear system of equations of the form  $Ax = b$ , where the coefficient matrix  $A \in \mathbb{R}^{n \times n}$  is nonsingular, large, sparse and nonsymmetric and also  $x, b \in \mathbb{R}^n$ . An  $ILU$  preconditioner  $M$  of this system is in the form of  $M = LDU \approx A$ . This preconditioner will change the original system to the left preconditioned system  $M^{-1}Ax = M^{-1}b$ . For a proper preconditioner, instead of solving the original system, it is better to solve the left preconditioned system by the Krylov subspace methods [3]. In [1], we have proposed an  $ILU$  preconditioner for system  $Ax = b$ . This preconditioner is termed the  $RIF - NS$  and has two left- and right-looking versions.

## 2 Pivoting strategy for the right-looking $RIF - NS$ preconditioner

Algorithm 1, uses the complete pivoting strategy to compute the right-looking version of  $RIF - NS$  preconditioner. Here we explain the step  $i$  of this algorithm. At the beginning of this step,  $\Pi = \Pi_{i-1} \dots \Pi_1$  and  $\Sigma = \Sigma_1 \dots \Sigma_{i-1}$  are the row and the column permutation matrices, respectively. For  $k < i$ , the matrices  $\Pi_k$  and  $\Sigma_k$  are the row and the column permutation matrices associated to step  $k$  of this algorithm. At the beginning of this step, the parameters  $m_i$ ,  $n_i$ ,  $iter$ ,  $satisfied\_p$  and  $satisfied\_q$  are initialized in line 3. At the end of this step,  $m_i$  and  $n_i$  will be the total number of row and column pivoting associated to step  $i$ . The parameter  $iter$  is used to compute the pivot entry in this step.  $satisfied\_p$  ( $satisfied\_q$ ) shows whether or not we need to the row (column) pivoting strategy. In line 7 of the algorithm, the vector  $(q_i^{(i-1)}, \dots, q_n^{(i-1)})$  is computed. Suppose that  $|q_k^{(i-1)}| = \max_{m \geq i+1} |q_m^{(i-1)}|$ . If the criterion  $|q_i^{(i-1)}| < \alpha |q_k^{(i-1)}|$  is satisfied for

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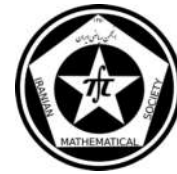
$\alpha \in (0, 1]$ , then the row pivoting strategy is applied in lines 8-11 of the algorithm. Suppose that  $|p_l^{(i-1)}| = \max_{m \geq i+1} |p_m^{(i-1)}|$ . If the criterion  $|p_i^{(i-1)}| < \alpha |p_l^{(i-1)}|$  is satisfied for an  $\alpha \in (0, 1]$ , then the column pivoting strategy is applied in lines 17-20 of the algorithm. After the column pivoting, *satisfied\_p* is set to *true* in line 22 and the algorithm will alternate between the row and the column pivoting. After the internal *while* loop, the pivot entry  $d_{ii}$  is set equal to  $q_i^{(i-1)}$ . In lines 25-28 of the algorithm, the  $i$ -th column of matrices  $W$  and  $L$ , and the  $i$ -th row of matrix  $U$  are computed.

### 3 Numerical results

In this section, we have formed 6 artificial linear systems where the coefficient matrices are downloaded from [2] and the exact solution of these systems is the vector  $[1, \dots, 1]^T$ . We have used two parameters 0.1 and 1.0 as  $\alpha$  to compute the right-looking version of  $RIF - NS$  preconditioner with complete pivoting strategy. We have used The command *bicgstab* in *Matlab* software to solve the original and the left preconditioned systems by the *BICGSTAB* method. The stopping criterion for all linear systems is satisfied when the relative residual is less than  $10^{-6}$ . We have considered the zero vector as the initial solution for all systems. The density of the preconditioners is defined as :

$$density = \frac{nnz(L) + nnz(U)}{nnz(A)},$$

where  $nnz(L)$ ,  $nnz(U)$  and  $nnz(A)$  are the number of nonzero entries of matrices  $L$ ,  $U$  and  $A$ . Table 1, shows the matrix properties and the information of *BICGSTAB* method to solve the original linear systems. In this table,  $n$  and  $nnz$  are the dimension and the number of nonzero entries of the matrix. In Tables 1 and 2, the parameters *it* and *flag* indicate the number of iterations and the status of the convergence. The parameter *iter* can be an integer+0.5 indicating convergence halfway through an iteration. When *flag* is equal to 0, it means that the method has been converged to the desired tolerance within the 2500 iterations. *flag* = 2 shows that the preconditioner is ill-conditioned and *flag* = 4 indicates that one of the scalar quantities calculating during the method became too small or too large to continue computing.



**Algorithm 1** Right-looking version of *RIF – NS* with complete pivoting

**Input:**  $A \in \mathbb{R}^{n \times n}$  and  $\tau_w, \tau_l, \tau_u \in (0, 1)$  be drop tolerances. **Output:**  $\Pi A \Sigma \approx M = LDU$

1.  $w_i^{(0)} = e_i, 1 \leq i \leq n$
2. **for**  $i = 1$  to  $n$  **do**
3.      $m_i = n_i = 0, iter = 0, satisfied\_p = satisfied\_q = false$
4.     **while** not *satisfied\_q* **do**
5.          $iter = iter + 1$
6.         If  $iter = 1$ , then set  $q_i^{(i-1)} = (w_i^{(i-1)})^T (\Pi A \Sigma) e_i$ . Otherwise set  $q_i^{(i-1)} = p_i^{(i-1)}$ .
7.          $q_j^{(i-1)} = (w_j^{(i-1)})^T (\Pi A \Sigma) e_i, i + 1 \leq j \leq n$
8.         **if**  $|q_i^{(i-1)}| < \alpha \max_{m \geq i+1} |q_m^{(i-1)}|$  **then**
9.              $m_i = m_i + 1, \pi_{m_i}^{(i-1)} = I_n$  and *satisfied\_p* = *false*
10.             Choose  $k$  such that  $|q_k^{(i-1)}| = \max_{m \geq i+1} |q_m^{(i-1)}|$ . Then, interchange columns  $i$  and  $k$  of  $W - I$  and rows  $i$  and  $k$  of  $\pi_{m_i}^{(i-1)}$  and  $L - I$ . Also interchange elements  $q_i^{(i-1)}$  and  $q_k^{(i-1)}$  and do the update  $\Pi = \pi_{m_i}^{(i-1)} \Pi$
11.         **end if**
12.         *satisfied\_q* = *true*
13.         **if** not *satisfied\_p* **then**
14.              $p_i^{(i-1)} = q_i^{(i-1)}$
15.              $p_j^{(i-1)} = (\Pi A \Sigma)_{ij}, i + 1 \leq j \leq n$ .
16.              $p_j^{(i-1)} = p_j^{(i-1)} - L_{ik} d_{kk} U_{kj}$  for  $k = 1$  to  $i - 1$  and  $j = i + 1$  to  $n$
17.             **if**  $|p_i^{(i-1)}| < \alpha \max_{m \geq i+1} |p_m^{(i-1)}|$  **then**
18.                  $n_i = n_i + 1, \sigma_{n_i}^{(i-1)} = I_n$  and *satisfied\_q* = *false*
19.                 Choose  $l$  such that  $|p_l^{(i-1)}| = \max_{m \geq i+1} |p_m^{(i-1)}|$ . Then, interchange columns  $i$  and  $l$  of  $\sigma_{n_i}^{(i-1)}$  and  $U - I$ . Also, interchange elements  $p_i^{(i-1)}$  and  $p_l^{(i-1)}$  and do the update  $\Sigma = \sigma_{n_i}^{(i-1)} \Sigma$
20.             **end if**
21.             **end if**
22.             *satisfied\_p* = *true*
23.         **end while**
24.          $d_{ii} = q_i^{(i-1)}$
25.         **for**  $j = i + 1$  to  $n$  **do**
26.              $w_j^{(i)} = w_j^{(i-1)} - (\frac{q_j^{(i-1)}}{d_{ii}}) w_i^{(i-1)}$  and for all  $l \leq i$ , if  $|w_{lj}^{(i)}| < \tau_w$ , then set  $w_{lj}^{(i)} = 0$
27.              $L_{ji} = \frac{q_j^{(i-1)}}{d_{ii}}, U_{ij} = \frac{p_j^{(i-1)}}{d_{ii}}$ . If  $|L_{ji}| < \tau_l$ , then set  $L_{ji} = 0$ . If  $|U_{ij}| < \tau_u$ , then set  $U_{ij} = 0$ .
28.         **end for**
29.         **end for**
30. Return  $L = (L_{ij})_{1 \leq i, j \leq n}, U = (U_{ij})_{1 \leq i, j \leq n}, D = diag(d_{ii})_{1 \leq i \leq n}, \Pi$  and  $\Sigma$ .

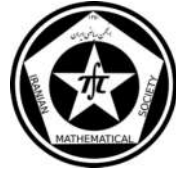
Table 1

Matrix	n	nnz	without preconditioner	
			it	flag
bwm200	200	796	109.5	0
str_400	363	3157	0	4
tols90	90	1746	28	4
str_0	363	2454	0	4
tub100	100	396	106.5	0
08blocks	300	592	1	4

In Table 2, the notation *RLRIF – NSP*( $\alpha$ ) refers to the right-looking version of *RIF – NS* preconditioner with complete pivoting strategy which is computed by using the parameter  $\alpha$ .

Table 2

Method	RLRIF-NSP(0.1)					RLRIF-NSP(1.0)					RLRIF-NS		
	density	Rpiv	Cpiv	iter	flag	density	Rpiv	Cpiv	iter	flag	density	iter	flag
bwm200	1.0012	0	0	23.5	0	1.2073	84	81	19.5	0	1.0012	23.5	0
str_400	0.5854	357	5	11	0	0.6097	383	57	0	2	5.5958	0	2
tols90	0.1523	18	0	2.5	0	0.4370	20	3	2.5	0	0.3070	12	0
str_0	0.5028	358	0	2	0	0.5676	362	29	2	0	3.9238	0	2
tub100	1.0050	0	0	8	0	1.1591	62	59	7.5	0	1.0050	8	0
08blocks	1.4797	292	0	1.5	0	118.3513	32	5	0	2	2	0.5	4



The  $RLRIF - NS$  is a notation for the right-looking version of  $RIF - NS$  preconditioner. The columns  $Rpiv$  and  $Cpiv$  show the total number of row and column pivoting. In this table, the information in the columns  $flag$  and  $iter$  associated to the three preconditioners indicate that for all of the matrices, one of the preconditioners  $RLRIF - NSP(1.0)$  or  $RLRIF - NSP(0.1)$  gives better results of the  $BICGSTAB$  method than the  $RLRIF - NS$  preconditioner. This means that the complete pivoting strategy with one of the values  $\alpha = 1.0$  or  $\alpha = 0.1$  has a good effect on the quality of the right-looking version of  $RIF - NS$  preconditioner.

## References

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