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# A BLOCK VERSION OF RIGHT-LOOKING A-BICONJUGATION PROCESS

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Abstract. In this paper, we present a block algorithm to factorize the inverse of a nonsymmetric and real matrix. This factorization has two unit upper triangular factors and a block diagonal matrix.

Keywords: Right-looking $A$-biconjugation process.

## 1. Introduction

Consider a nonsymmetric, real, square and invertible matrix $A$. In [1], the authors have used an algorithm called right-looking $A$-biconjugation to factorize the inverse of matrix $A$ as the following

$$A^{-1} = ZD^{-1}W^T, \tag{1.1}$$

where $Z$ and $W$ are unit upper triangular matrices and $D$ is a diagonal matrix.

Suppose that matrix $A$ has the factorization

$$A = LDU,$$

where $L$ and $U^T$ are unit upper triangular matrices and $D$ is a diagonal matrix. This factorization can be computed by the Gaussian elimination process [3]. At step $i$ of this process, relation

$$A = \underbrace{\begin{bmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && \ddots &&&& \\ &&& 1 &&& \\ g_1 & g_2 && g_{i-1} & 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} D_{11} &&& \\ & \ddots && \\ && D_{i-1,i-1} & \\ &&& (S^{(i-1)})_{j,k\geq i} \end{bmatrix}}_{S^{(i-1)}} \times \underbrace{\begin{bmatrix} 1 &&&& h_1 && \\ & 1 &&& h_2 && \\ && \ddots &&&& \\ &&& 1 & h_{i-1} && \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix}}_{U}. \tag{1.2}$$

holds. In (1.2), the submatrix $(S^{(i-1)})_{j,k\geq i}$ is termed the Schur-Complement matrix. In [2], the author has presented a block version of this algorithm. In this

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**Algorithm 1 ( A block format of right-looking A-biconjugation process )**

**Input**: $A \in \mathbb{R}^{n \times n}$ a nonsymmetric matrix.
**Output**: $A = ZD^{-1}W^T$. $Z$ and $W$ are unit upper triangular and $D$ is a block diagonal .

1. $status(i) = 0,\ 1 \le i \le n$
2. $i = 1$
3. **while** $i < n$ **do**
4. **for** $j = i$ to $n$ **do**
5. $S_{ij}^{(i-1)} = (w_i^{(i-1)})^T A z_j^{(i-1)}$
6. $S_{i+1,j}^{(i-1)} = (w_{i+1}^{(i-1)})^T A z_j^{(i-1)}$
7. **end for**
8. **for** $j = i+2$ to $n$ **do**
9. $S_{ji}^{(i-1)} = (w_j^{(i-1)})^T A z_i^{(i-1)}$
10. $S_{j,i+1}^{(i-1)} = (w_j^{(i-1)})^T A z_{i+1}^{(i-1)}$
11. **end for**
12. $v_i = max\{\frac{1}{|S_{ii}^{(i-1)}|} \sum_{j=i+1}^n |S_{ij}^{(i-1)}|, \frac{1}{|S_{ii}^{(i-1)}|} \sum_{j=i+1}^n |S_{ij}^{(i-1)}|\}$
13. $w_i^1 = \sum_{j=i+2}^n || \begin{bmatrix} S_{ii}^{(i-1)} & S_{i,i+1}^{(i-1)} \\ S_{i+1,i}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \end{bmatrix}^{-1} \begin{pmatrix} S_{ij}^{(i-1)} \\ S_{i+1,j}^{(i-1)} \end{pmatrix} ||_\infty$
14. $w_i^2 = \sum_{j=i+2}^n || \begin{pmatrix} S_{ji}^{(i-1)} & S_{j,i+1}^{(i-1)} \end{pmatrix} \begin{bmatrix} S_{ii}^{(i-1)} & S_{i,i+1}^{(i-1)} \\ S_{i+1,i}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \end{bmatrix}^{-1} ||_\infty$
15. $w_i = \max\{w_i^1, w_i^2\}$
16. **if** $v_i < w_i$ **then**
17. $D_{ii} = S_{ii}^{(i-1)}$
18. $status(i) = 1$
19. Column-Const $(Z\ ,A\ ,\ D\ ,\ i\ ,\ status\ )$
20. Column-Const $(W\ ,\ A^T\ ,\ D^T\ ,\ i\ ,\ status\ )$
21. $i = i + 1$
22. **end if**
23. **if** $v_i \ge w_i$ **then**
24. $D_{i:i+1,i:i+1} = \begin{bmatrix} S_{ii}^{(i-1)} & S_{i,i+1}^{(i-1)} \\ S_{i+1,i}^{(i-1)} & S_{i+1,i+1}^{(i-1)} \end{bmatrix}$
25. $z_{i+1}^{(i)} = z_{i+1}^{(i-1)}$
26. $w_{i+1}^{(i)} = w_{i+1}^{(i-1)}$
27. $status(i) = 2$
28. $Column - Const(Z, A, D, i, status)$
29. $Column - Const(W, A^T, D^T, i, status)$
30. $i = i + 2$
31. **end if**
32. **end while**
33. **if** $status(n-1) = 0 \ || \ status(n-1) = 1$ **then**
34. $D_{nn} = (w_n^{(i-1)})^T A z_n^{(i-1)}$
35. **end if**
36. Return $Z = [z_1^{(0)}, z_2^{(1)}, \cdots, z_n^{(n-1)}]$, $W = [w_1^{(0)}, w_2^{(1)}, \cdots, w_n^{(n-1)}]$ and $D$

block format, $L$ and $U^T$ are again unit upper triangular and $D$ is a block diagonal matrix. The diagonal blocks of $D$ are of order $1 \times 1$ or $2 \times 2$. In this paper we use the method in [2] and present a block format for the right-looking $A$-biconjugation process.

## 2. Block algorithm to factorize $A^{-1}$

Algorithm 1 is the block version of $A$-biconjugation process. It computes the factorization in (1.1). $Z$ and $W$ are unit upper triangular while $D$ is a block diagonal matrix with blocks of order $1 \times 1$ or $2 \times 2$. Here we explian step $i$ of this algorithm. Before this step, we first initialize the array *status*. Then $i$ is set to 1 and we enter the internal *while* loop. In lines 4-11 we obtain the first two rows and columns of the Schur-Complement matrix $(S^{(i-1)})_{j,k \ge i}$ in (1.2). In line 12, the

**Algorithm 2** ( **Column construction** )

$Column - Const(Z, A, D, i, status)$
**if** $status(i) = 1$ **then**
  **for** $j = i + 1$ to $n$ **do**
    $z_j^{(i)} = z_j^{(i-1)} - z_i^{(i-1)} \frac{1}{D_{ii}} A_{i,:} z_j^{(i-1)}$
  **end for**
**end if**
**if** $status(i) = 2$ **then**
  **for** $j = i + 2$ to $n$ **do**
    $k = i + status(i) - 1$
    $z_j^{(k)} = z_j^{(i-1)} - [z_i^{(i-1)} \; z_{i+1}^{(i)}] D^{-1}_{i:i+1,i:i+1} A_{i:i+1} z_j^{(i-1)}$
  **end for**
**end if**

parameter $v_i$ is defined. Then, in lines 13 and 14 we compute the two parameters $w_i^1$ and $w_i^2$ and in line 15, $w_i$ is considered as the maximum of these two values. After that, we check in line 16 if $v_i < w_i$. In this case, $D_{ii}$ is considered as $S_{ii}^{(i-1)}$, $status(i)$ is set to 1 and we call Algorithm 2 in lines 19 and 20 to update the columns $(i+1)$st to $n$ of matrices $Z$ and $W$. At the end, $i$ is incremented by 1. If $v_i \geq w_i$, then in lines 24-30 we define a $2 \times 2$ block diagonal, the final $(i+1)$st columns of $W$ and $Z$ are introduced, $status(i)$ is set to 2 and we call Algorithm 2 twice in lines 28 and 29. Finally, $i$ is incerемented by 2. After the *while* loop, if $status(n-1)$ is equal to 0 or 1, then the last $1 \times 1$ diagonal entry is computed.

## 3. Numerical example

We have implemented Algorithms 1 and 2 in MATLAB. Consider the nonsymmetric and invertible matrix A as

$$A = \begin{bmatrix} 3 & 3 & 1 & 1 & 4 & 2 & 2 \\ 3 & 2 & 2 & 1 & 4 & 3 & 3 \\ 4 & 4 & 2 & 2 & 2 & 3 & 1 \\ 4 & 4 & 1 & 2 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 2 & 1 & 4 \\ 2 & 3 & 1 & 2 & 2 & 2 & 1 \\ 4 & 3 & 1 & 1 & 3 & 2 & 4 \end{bmatrix}.$$

If we provide this matrix as the input argument of Algorithm 1, then the computed $Z$, $D$ and $W$ are

$$Z = \begin{bmatrix} 1.0000 & 0 & -1.3333 & 1.0000 & -8.0000 & 0.1154 & -2.0385 \\ 0 & 1.0000 & 1.0000 & -1.0000 & 5.0000 & -1.1923 & 2.7308 \\ 0 & 0 & 1.0000 & -1.0000 & 5.0000 & -2.1923 & 1.7308 \\ 0 & 0 & 0 & 1.0000 & 0 & 2.6538 & -2.8846 \\ 0 & 0 & 0 & 0 & 1.0000 & 0.1923 & -0.7308 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$D = \begin{bmatrix} 3.0000 & 3.0000 & 0 & 0 & 0 & 0 & 0 \\ 3.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & -6.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.1538 & -0.3846 \\ 0 & 0 & 0 & 0 & 0 & -0.0769 & 0.6923 \end{bmatrix}$$

$$W = \begin{bmatrix} 1.0000 & 0 & -1.3333 & -2.0000 & 5.0000 & -1.3077 & 0.1538 \\ 0 & 1.0000 & 0 & 0 & 0 & 1.0000 & -1.0000 \\ 0 & 0 & 1.0000 & 0.5000 & -4.5000 & 0.0769 & 0.4615 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & -1.0000 \\ 0 & 0 & 0 & 0 & 1.0000 & -0.4615 & 0.2308 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

## References

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